Mathematical Morphology: an Algebraic Approach

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Mathematical morphology is a theory on morphological transformations which form the basic components for a number of algorithms in quantitative image analysis. In this paper we present an overview of the basic principles of mathematical morphology, and initiate a generalization of the theory by taking the object space to be an arbitrary complete lattice.

1. Principles of mathematical morphology

1.1. Introduction

A person who comes into touch with image processing for the very first time will probably be overwhelmed by the enormous amount of literature that appears every year, and it is not unlikely that he or she will be deterred by the dispersion which characterizes the field. A first branch, which is beyond the scope of this paper, originates from classical signal analysis, and its basic tools are convolution and (Fourier, Karhunen-Loeve) filtering methods. Most of the operations are linear and sometimes even reversible, which means that its performance is not attended with loss of information. For a rather complete account of this approach we refer to ROSENFELD and KAK [11]. A second branch in image processing is formed by mathematical morphology, a somewhat axiomatic theory containing elements of integral geometry, stereometry and stochastic geometry.

Essentially, mathematical morphology is a theory on morphological transformations and functionals, which, if chosen properly, make it possible to measure useful geometric features of images. The main body of the theory was developed at the Centre of the Paris School of Mines at Fontainebleau in France, and its success is due in part to the fact that the theoretical research kept pace with the development of an image analysis system, called the 'texture analyser'. The books of MATHERON [9] and SERRA [12] (see also [1,3]) provide a complete overview of the theory of mathematical morphology, the main idea of which is captured by the following quotation from the Preface of [12]:

'The notion of a geometrical structure, or texture, is not purely objective. It does not exist in the phenomenon itself, nor in the observer, but somewhere in between the two. Mathematical morphology quantifies this intuition by introducing the concept of structuring elements. Chosen by the morphologist, they interact with the object under study, modifying its shape and reducing it to a sort of caricature which is more expressive than the actual initial phenomenon...'

A morphological transformation of an image (a subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$) is obtained by taking in a prescribed manner unions and intersections of a number of translates of this set and its complement. The collection of translation vectors involved constitutes the so-called structuring element. In practice one can reveal certain geometrical information about objects by sequential application of morphological transformations involving cleverly chosen structuring elements: it is clear that the number of possibilities is unlimited.

An important feature of (nontrivial) morphological transformations is their irreversibility: the transformed image contains less information than the original one. Or in mathematical terms: morphological transformations are not injective.

In the discrete case morphological transformations bear much resemblance to cellular automata (or cellular logic) transformations. Such transformations are performed by giving each pixel a new state depending on its old state and the old state of its neighbours: see [4,10]. An implicit consequence of the specific structure of a morphological transformation, which is of great practical value, is that one can use the build-in parallelism of the computer.

This paper consists of two parts. In the first part we survey some of the basic theory, whereas in the second part we indicate how this theory can be generalized to complete lattices. In the following section we present the basic transformations of mathematical morphology, namely dilation, erosion, closing and opening. The step from the continuous to the discrete space, involving the digitalization of images, can be justified if one can supply the continuous object space (whose elements are sets) with a topology. The introduction of a topology also enables one to prove robustness of transformations. In Section 1.3 we present such a topology. At that place we also discuss the basic principles, which, according to Serra's philosophy, define the morphological transformations. These principles include translation invariance and semi-continuity. At the end of Section 1.3, we formulate some mathematical questions raised by these principles. Together with the inborn impulse of any mathematician to generalize whatever he can lay hands on, these questions have been our main motivation to strive for a more axiomatic algebraic approach.

Such an algebraic approach is initiated in Part 2. There the basic assumption is that the underlying object space forms a complete lattice. In Section 2.1 we survey the relevant results of lattice theory. In Section 2.2 we give an abstract definition of dilation and erosion: this definition depends on the choice of a commutative automorphism group on the lattice (being the generalization of the translation group on $\mathbb{R}^n$ or $\mathbb{Z}^n$). Under some extra assumptions we can give a complete characterization of dilations and erosions. MATHERON [9] has proved that every increasing translation invariant transformation can be written as an intersection of dilations, or equivalently, as a union of erosions. In Section 2.3 we prove an abstract version of this theorem. Finally, in Section 2.4, we speculate about what has to be done in the future.

1.2. Dilation and erosion, closing and opening

Throughout this section, let E be the Euclidean space $\mathbb{R}^n$ or the discrete space $\mathbb{Z}^n$. Essential is that E is a commutative group. Let $\mathcal{P}(E)$ be the space of all subsets of E. A binary image can be represented by a subset X of E. We call X the object and $\mathcal{P}(E)$ the object space. If $X \subset E$ and $h \in E$ then we denote by X_h the translate of X along h:

$$X_h = \{x + h : x \in X\}.$$

If $X, Y \subset E$ then we say that X hits Y, $X \Uparrow Y$, if $X \cap Y \neq \varnothing$. Let A be an arbitrary subset of E. The *dilation* of a set X by the element A is defined by

$$X \dot{\oplus} A = \{h \in E : A_h \Uparrow X\}.$$

The *erosion* of X by A is defined by

$$X \dot{\ominus} A = \{h \in E : A_h \subset X\}.$$

We call A the *structuring element*. It is an easy exercise to show that the dilation of an image gives the same result as the erosion of its background, i.e.

$$(X \check{\oplus} A)^c = X^c \check{\ominus} A.$$

Here X^c denotes the complement of X. We say that dilation and erosion are complementary (or dual) operations. Let the Minkowski addition $\oplus$ and subtraction $\ominus$ of two sets $X, Y \subset E$ respectively be given by

$$X \oplus Y = \bigcup_{y \in Y} X_y$$

$$X \ominus Y = \bigcap_{y \in Y} X_y.$$

Then we have the following relationships:

$$X \check{\oplus} A = X \oplus \check{A}$$

$$X \check{\ominus} A = X \ominus \check{A},$$

where $\check{A} = -A = \{-a : a \in A\}$. The incredulous reader may verify this. Typical properties of dilation are

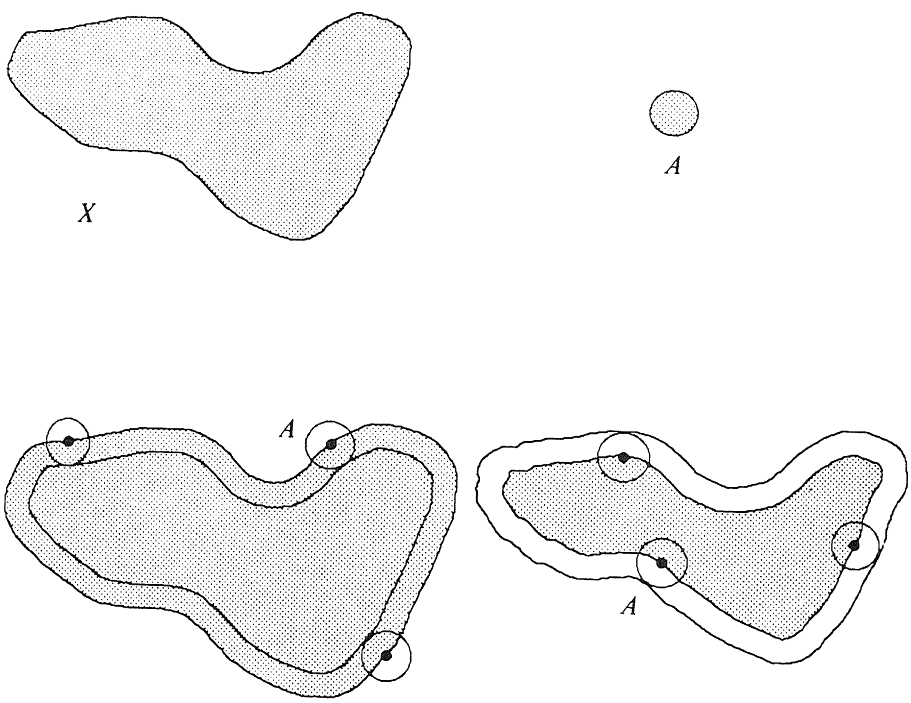
(i) $(X \dot{\oplus} A)_h = X_h \dot{\oplus} A$,

(ii) $(\bigcup_{i \in I} X_i) \dot{\oplus} A = \bigcup_{i \in I} (X_i \dot{\oplus} A)$

where I is an arbitrary finite or infinite index set, and $X_i \subset E$, $i \in I$. Thus dilation is distributive with respect to union and invariant under translation. Similar properties hold for erosion

(i) $(X \dot{\ominus} A)_h = X_h \dot{\ominus} A$.

(ii) $(\bigcap_{i \in I} X_i) \dot{\ominus} A = \bigcap_{i \in I} (X_i \dot{\ominus} A)$.



dilation of X by A erosion of X by A

FIGURE 1. Dilation and erosion in the continuous case

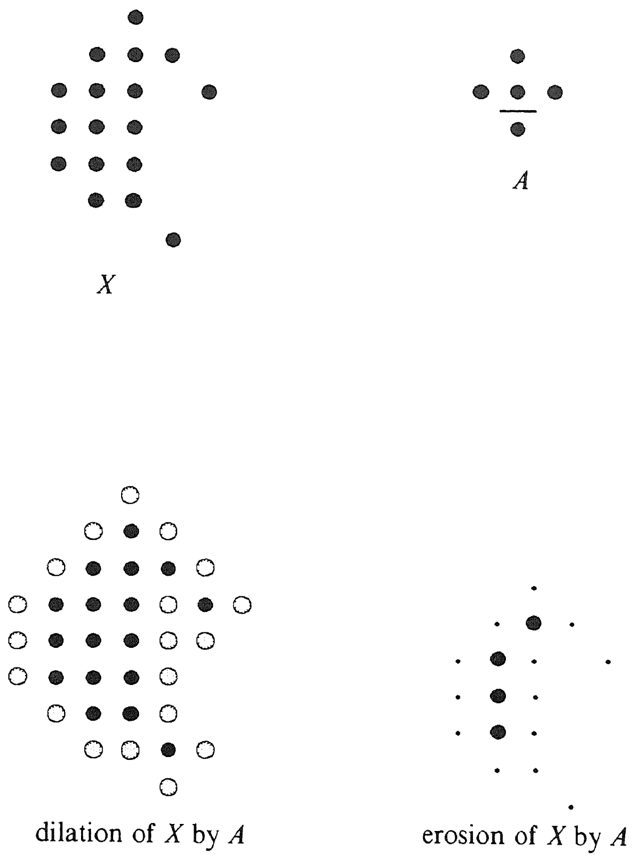


FIGURE 2. Dilation and erosion in the discrete case

● points which belong to X

◍ points which belong to $X\oplus\check{A}$ but not to X

· points which belong to X but not to $X\ominus\check{A}$

The underlining in A denotes the location of the origin.

One can easily prove the following algebraic relations:

$$(X \oplus A) \oplus B = X \oplus (A \oplus B)$$

$$(X \ominus A) \ominus B = X \ominus (A \oplus B)$$

$$X \oplus (A \cup B) = (X \oplus A) \cup (X \oplus B)$$

$$X \ominus (A \cup B) = (X \ominus A) \cap (X \ominus B).$$

These relations have the important practical implication that dilations and erosions with a structuring element which is too large to be handled by the hardware at one stage can be decomposed. Although it is true that dilation and erosion have a very simple algebraic structure, their importance is enormous. Perhaps this is most clearly illustrated by a theorem of MATHERON [9] which we state below. But first we give some definitions.

Let Ψ be a mapping from the object space $\mathcal{P}(E)$ into itself. We say that Ψ is *increasing* if

$$X \subset Y \Rightarrow \Psi(X) \subset \Psi(Y).$$

Note that dilation and erosion are increasing transformations. We call Ψ *translation invariant* if

$$\Psi(X_h) = (\Psi(X))_h,$$

for every $X \subset E$ and $h \in E$. The complementary (or dual) mapping Ψ^* of Ψ is defined by

$$\Psi^*(X) = (\Psi(X^c))^c.$$

The *kernel* $\mathcal{V}$ of a mapping Ψ is defined by

$$\mathcal{V} = \{A \subset E : 0 \in \Psi(A)\}.$$

The kernel of the dual mapping Ψ^* is denoted by $\mathcal{V}^*$.

MATHERON'S THEOREM. *Let $\Psi: \mathcal{P}(E) \to \mathcal{P}(E)$ be an increasing, translation invariant mapping. Then*

$$\Psi(X) = \bigcup_{A \in \mathcal{V}} (X \ominus A) = \bigcap_{A \in \mathcal{V}^*} (X \oplus A),$$

for every $X \subset E$.

Note that the second equality follows from the first by duality. In Section 2.3 we shall prove an abstract version of Matheron's Theorem.

Two important increasing, translation invariant transformations on $\mathcal{P}(E)$ are the *closing* and the *opening*. The closing and the opening of a set X by a structuring element A are respectively defined by

$$X^A = (X \oplus A) \ominus A$$

$$X_A = (X \ominus A) \oplus A.$$

Closing and opening are complementary transformations. Some straightforward manipulations show that for every $X \subset E$:

$$X_A \subset X \subset X^A,$$

i.e., closing is an *extensive* operation whereas opening is *anti-extensive*. Furthermore, both operations are *idempotent*:

$$(X^A)^A = X^A, \ (X_A)_A = X_A.$$

Morphological transformations which are increasing and idempotent are sometimes called *morphological filters* or *M-filters*. Note the analogy with the ideal band-pass filter from classical signal analysis.

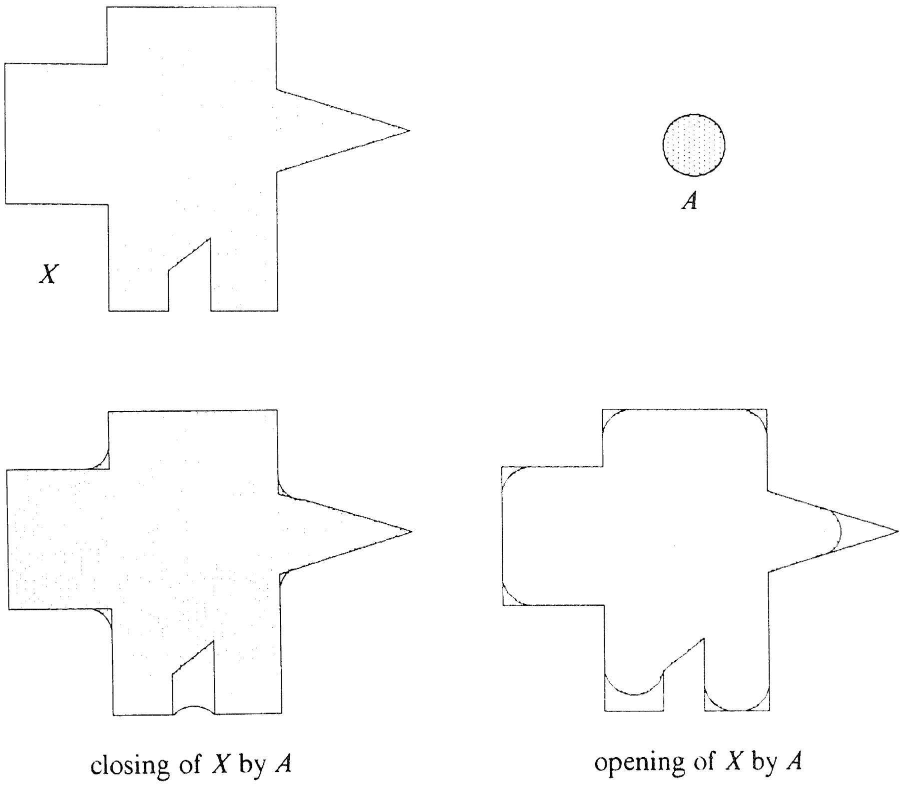


FIGURE 3. Closing and opening in the continuous case

We conclude this section by indicating an application of the opening. This operation makes it possible to define *size distributions*. This goes roughly as follows. An object built up of several smaller and larger isolated grains is put through a sequence of smaller and smaller sieves. Then a size distribution of X is given by the function $r \rightarrow \text{area}(X_{rA})$, where A is a compact convex structuring element (its shape may be chosen according to the shape of the grains), and $r > 0$ is a measure for the width of the sieve.

So far, the objects under study are considered as subsets of the continuous Euclidean space $\mathbb{R}^n$, or the discrete space $\mathbb{Z}^n$. Eventually, one is also interested in grey-valued images. Although such objects do not a priori fit into the framework, it is possible to extend the theory to account for them as well. There are at least two ways to do this. The first way is to represent each grey-valued image by a continuum of sets, the so-called cross sections. To every cross section one can apply the original morphological transformation, thus obtaining a new continuum of sets representing the transformed grey-valued image. The second way is to represent an image by its umbra (the graph together with all points in its shadow) which is a set again. To this set one can apply a morphological transformation yielding an umbra again, and from this the transformed grey-valued image is easily obtained. This is all we are going to say about this matter, and we refer the interested reader to Serra's book [12] and to a paper by STERNBERG [13]. For the rest of this paper we shall restrict to binary (i.e. black-white) images.

1.3. Morphological transformations

From Matheron's theorem we learned that dilation and erosion are very important transformations, since they are the building blocks of all increasing translation invariant transformations on $\mathcal{P}(E)$. A moment of reflection tells us that they also constitute the basis for all decreasing, translation invariant mappings. Namely, if $X \rightarrow \Phi(X)$ is decreasing (i.e., $X \subset Y \Rightarrow \Phi(Y) \subset \Phi(X)$), then the mapping $X \rightarrow \Phi(X^c)$ is increasing and Matheron's theorem yields that

$$\Phi(X) = \bigcup_{A \in \mathcal{W}} (X^c \dot{\ominus} A),$$

where $\mathcal{W} = \{A \subset E : 0 \in \Phi(A^c)\}$.

An example of a transformation which needs to be neither increasing nor decreasing is the *hit-or-miss transformation*, which can e.g. be used to detect corner points of objects. Here the structuring element consists of two components A and B. Its definition goes as follows:

$$X \circledast [A,B] = \{h \in E : A_h \subset X \text{ and } B_h \subset X^c\} = (X \dot{\ominus} A) \cap (X \dot{\oplus} B)^c.$$

We present an example in Figure 4.

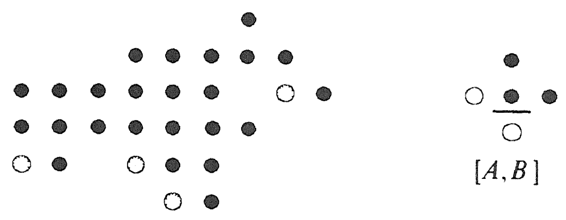


FIGURE 4. The hit-or-miss transformation can be used to detect corner points. ● and ◎ belong to X and ○ belongs to $X \circledast [A,B]$. The structuring element consists of a component A given by ● and B given by ○.

As a next step one can define the *thinning* $X \rightarrow X \setminus X \circledast [A,B]$ and the *thickening* $X \rightarrow X \cup (X \circledast [A,B])$. The thinning and thickening operation form the basis for a whole collection of algorithms which transform sets into figures with exotic names like *skeleton, homotopic pruning, skiz*, and *pseudo-convex hull*. We refer the inquisitive reader to chapter XI of Serra's book. At this place it is important to mention that Serra works on the hexagonal grid, and that he chooses the structuring elements accordingly.

The hit-or-miss transformation also forms the foundation for the definition of a topology on a space of subsets of E. A topology is indispensable to estimate errors committed in digitalizing images and to prove (or disprove) robustness of certain image transformations. Around 1974, G. MATHERON [9] and D.G. KENDALL [7], independently of each other, laid the foundations for a general theory of random sets, and it is not too surprising that these breakthroughs have had a strong impact on the development of mathematical morphology. We give a very short outline of Matheron's approach. Also see [1].

Let E be a topological space which is locally compact, Hausdorff, and separable (i.e., E admits a countable base). Of course, the example we have at the back of our mind is $E = \mathbb{R}^n$. We shall introduce a topology, the so-called hit-or-miss topology, on $\mathfrak{F}(E)$, the space of all closed subsets of E, but we do not refrain from noting that we might as well have chosen the open subsets. Let $K \subset E$ be compact and $G \subset E$ open. We define

$$\mathfrak{F}^K = \{X \in \mathfrak{F}(E) \colon X \cap K = \varnothing\}$$

$$\mathfrak{F}_G = \{X \in \mathfrak{F}(E) \colon X \cap G \neq \varnothing\}.$$

The hit-or-miss topology on $\mathfrak{F}(E)$ is defined by the base elements $\mathfrak{F}^K \cap \mathfrak{F}_{G_1} \cap \cdots \cap \mathfrak{F}_{G_m}$, where K is compact, and G_i is open, $i = 1, \ldots, m$. In other words, the sets $\mathfrak{F}^K$ and $\mathfrak{F}_G$ form a subbase for the hit-or-miss topology.

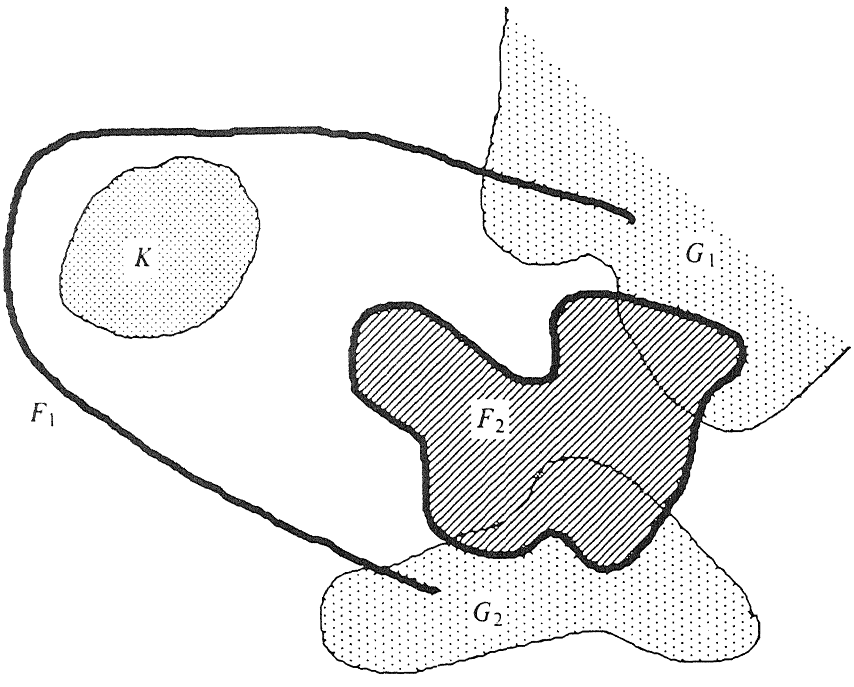


FIGURE 5. F_1 and F_2 both belong to the base element $\mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2}$

The space $\mathcal{F}(E)$ equipped with the hit-or-miss topology is compact, Hausdorff, and separable. Note that E is not required to be compact but only locally compact: see MATHERON [9, Theorem 1.2.1]. A random closed set is by definition a random element of $\mathcal{F}(E)$ with the Borel σ-algebra. In fact, every random closed set is specified by the probability distribution $p[K \cap X = \varnothing]$ where K ranges over all compact subsets of E.

Let ψ be a mapping from an arbitrary topological space S into $\mathcal{F}(E)$. Then ψ is upper-semi-continuous (u.s.c.) if for any compact set $K \subset E$, the set $\psi^{-1}(\mathcal{F}^K)$ is open in S. Analogously, ψ is lower-semi-continuous (l.s.c.) if for any open set $G \subset E$, the set $\psi^{-1}(\mathcal{F}_G)$ is open in S. If the topological space S admits a countable base, in particular if $S = \mathcal{F}(E)$, then there exist some easy criteria for upper- and lower-semi-continuity: see [9], [12]. For the basic transformations of mathematical morphology, MATHERON [9] has obtained the following continuity results:

(i) $X \to X \oplus A$ is continuous on $\mathcal{F}(E)$ if A is compact

(ii) $X \to X \ominus A$, $X \to X^A$, and $X \to X_A$ are upper-semi-continuous if A is compact.

Actually, Matheron proved a much stronger result.

In Chapter I of his book [12], Serra treats at length four principles which, according to his philosophy, every transformation has to satisfy in order to get the predicate 'morphological'. These principles, which we discuss below, are unmistakably inspired by practical considerations.

The first principle, concerning translation invariance, excludes transformations which require knowledge of the position of the object of interest. In mathematical terms:

$$\Psi(X_h) = (\Psi(X))_h. \qquad \text{(I)}$$

Frequently, an object has to be magnified or reduced before one can work with it. For transformations one wants to apply, this means that they have to be compatible under change of scale. Denoting the transformation by Ψ_λ, where λ is the scale parameter, we can write the second principle abstractly as

$$\Psi_\lambda(\lambda X) = \lambda\Psi_1(X), \qquad \text{(II)}$$

where $\lambda X = \{\lambda x : x \in X\}$.

The third principle says that local knowledge of the object is sufficient to obtain local knowledge about the transformed image:

$$\forall_X \forall_{\text{bounded } Z'} \exists_{\text{bounded } Z} : [\Psi(X \cap Z)] \cap Z' = \Psi(X) \cap Z'. \qquad \text{(III)}$$

Note that this definition allows that Z depends on X: in practical cases this will almost never occur.

The last principle says something about stability of the transformation:

Ψ is semi-continuous with respect to the hit-or-miss topology. (IV)

Note that this last principle implicitly assumes that Ψ maps closed sets on closed sets.

The basic transformations dilation, erosion, closing and opening indeed satisfy the four principles if the structuring element is compact and nonempty. As far as the applications are concerned, these principles are quite satisfactory. But they also evoke a number of questions in a theoretician's mind. Let us state some of them. (1) Is it possible to give a complete characterization of all morphological transformations? Matheron's theorem only gives a partial answer to this question. (2) As we already mentioned, the fourth principle includes the assumption that the object space should be $\mathcal{F}(E)$ instead of $\mathcal{P}(E)$. But the algebraic structure of these two spaces are completely different (see Part 2 below). For example, we do not have a natural complement on $\mathcal{F}(E)$, which means in particular that the definition of the hit-or-miss transformation needs to be adapted. (3) The specific structure of dilation and erosion shows that translation plays a very special role. Why is this? Can this role be assigned to another group operation on E, rotation for instance? Note that rotation invariance is not included in the four principles.

These and other questions have motivated us to look for a more abstract approach, not so much because we expect new applications, but merely because we hope that such an approach gives a better understanding.

2.1. Some basic results from lattice theory

In this preliminary section we survey some of the basic results on lattices. For a complete account of the theory we refer to the monographs of BIRKHOFF [2] and GRÄTZER [6].

A set L with a partial ordering relation $\leqslant$ is called a *lattice* if for any finite nonempty subset K of L the *supremum* $\vee K$ and *infimum* $\wedge K$ exist. Recall that $a \in L$ is called the supremum of K if $x \leqslant a$ for every $x \in K$ and if $a \leqslant a'$ for any other such element a'. A similar definition holds for the infimum. We shall write $x \vee y$ instead of $\vee\{x,y\}$ and $x \wedge y$ instead of $\wedge\{x,y\}$. It is easily seen that

$$x \leqslant y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y. \tag{2.1}$$

We write $x < y$ if $x \leqslant y$ and $x \neq y$. A lattice L can contain at most one element a which satisfies $a \leqslant x$ for all $x \in L$. If such an element exists we denote it by 0 and call it the *zero* of L. Similarly, there can exist at most one element b such that $x \leqslant b$ for all $x \in L$. Such an element, if present, is called the *unit* of L and is denoted by 1. A lattice with a zero and a unit is called *bounded*. The lattice is called *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2.2a}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \tag{2.2.b}$$

for every $x,y,z \in L$. Let L be a bounded lattice. We say that x possesses a *complement* y if

$$x \wedge y = 0, \ x \vee y = 1. \tag{2.3}$$

The bounded lattice L is called *complemented* if any of its elements has a complement. It is an easy exercise to show that in a bounded distributive lattice an element x can have at most one complement which is then denoted by x^*.

DE MORGAN'S IDENTITIES. *Let x,y be elements of the bounded, distributive lattice L with complements x^* and y^* respectively. Then $x \vee y$ and $x \wedge y$ possess complements as well, and*

$$(x \vee y)^* = x^* \wedge y^*$$

$$(x \wedge y)^* = x^* \vee y^*.$$

A complemented, distributive lattice is called a *Boolean lattice*. Every Boolean lattice can be considered as an algebra with the binary operations $\vee$ and $\wedge$, and the unary operation $*$. Considered this way, L is called a *Boolean algebra*. In a number of cases the lattice L is only 'half-complemented' in the sense that only one of the relations in (2.3) is satisfied. A *Brouwerian lattice* is a lattice L in which for every couple, $a,b \in L$ the set $\{x : a \wedge x \leqslant b\}$ contains a greatest element $b:a$, the *relative pseudo-complement of a in b*: below we shall present an example. If L is a Boolean lattice, then, of course, $b:a = b \vee a^*$. In a

Brouwerian lattice with a zero, the element $x^* = 0{:}x$ is called the *pseudo-complement* of x. Note that, by definition, x^* is uniquely defined. A theorem in [2] says that every Brouwerian lattice is distributive. It is not hard to figure out how dual Brouwerian lattices should be defined.

A lattice L is called *complete* if any subset K (finite or infinite) has a supremum and an infimum. If L is a nonempty complete lattice then one gets, by taking $K = L$, that L has a zero and a unit. It is easily deduced from (2.2) that in any distributive lattice L the relations

$$x\wedge(\vee_{i\in I}x_i) = \vee_{i\in I}(x\wedge x_i) \tag{2.4a}$$

$$x\vee(\wedge_{i\in I}x_i) = \wedge_{i\in I}(x\vee x_i) \tag{2.4b}$$

are valid for any finite index set I. In a complete Boolean lattice these relations hold for any infinite index set as well. A lattice in which (2.4a) is valid for an arbitrary index set is called *infinite-supremum-distributive*, whereas the lattice is called *infinite-infimum-distributive* if (2.4b) holds. It is relatively easy to show that a complete lattice is Brouwerian if and only if it is infinite-supremum-distributive, and in that case $b{:}a = \vee\{x : a\wedge x \leqslant b\}$.

Let L be a lattice with a zero. An element $\xi \in L$ is called an *atom* if $x < \xi$ implies that $x = 0$. Analogously, an element ξ' of a lattice with a unit is called a *dual atom* if $\xi' < x$ implies that $x = 1$. Atoms are denoted by Greek symbols and dual atoms by Greek symbols with a prime. We denote the set of all atoms by Λ. An *atomic lattice* is a lattice in which every element is the supremum of the atoms it dominates, i.e.,

$$x = \vee_{\xi\leqslant x}\xi.$$

Similarly, we define dually atomic lattices.

The reader who wishes to know more about lattices and the relation with set theory may consult [2,5,6,8]. For those who had enough, we present some examples. It goes without saying that our choice is highly influenced by the application we have in mind.

EXAMPLES

(a) Let E be some nonempty set. Then $\mathcal{P}(E)$ is a complete lattice with the partial ordering: $X \leqslant Y$ if $X \subset Y$, i.e., X is included in Y. The supremum and infimum correspond to the union and intersection respectively. With the set complement $\mathcal{P}(E)$ becomes a Boolean lattice. Moreover, $\mathcal{P}(E)$ is atomic where the atoms are of the form $\{e\}$, where $e \in E$. At this point we mention the following important general result. Every complete, atomic Boolean lattice L is isomorphic to the field $\mathcal{P}(\Lambda)$, where Λ is the set of all atoms of L.

(b) If E is a nonempty topological space, then we denote by $\mathcal{F}(E)$ the space of all closed subsets of E. If we define (a 'bar' denoting closure)

$$\wedge_{i\in I}X_i = \bigcap_{i\in I}X_i$$

$$\bigvee_{i \in I} X_i = \overline{\bigcup_{i \in I} X_i},$$

for an arbitrary index set I and arbitrary elements $X_i \in \mathcal{F}(E)$, then $\mathcal{F}(E)$ is a complete, distributive lattice which is infinite-infimum-distributive. Moreover, if E is a T_1-space (i.e., every singleton $\{e\}$ with $e \in E$ is closed), then $\mathcal{F}(E)$ is atomic. In this case, BIRKHOFF [2] calls $\mathcal{F}(E)$ a T_1-lattice.

The space $\mathcal{G}(E)$ of all open subsets of the topological space E is a complete, distributive lattice which is infinite-supremum-distributive, hence $\mathcal{G}(E)$ is a Brouwerian lattice with pseudo-complement $X^* = (\overline{X})^c$. If $X = X^{**}$, then we call X a *regular open set*. We leave it as an exercise to the reader to verify that the space of all regular open subsets forms a complete Boolean lattice.

(c) As a final example we mention the lattice consisting of all functions f mapping a set E into the closed interval $[0,1]$, with the pointwise ordering:

$$f \leq g \Leftrightarrow f(x) \leq g(x), \quad \forall_{x \in E}.$$

Note that this lattice is relevant in the context of grey-value images. The supremum and infimum are respectively defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$. It is obvious that this lattice is complete and distributive. Furthermore, it is worth noticing that the lattice of example (a) lies embedded in the present one, where the embedding operation is given by $X \rightarrow \mathbf{1}_X$, $X \subset E$. Here $\mathbf{1}_X$ is the characteristic function corresponding to the set X.

The remainder of this section is devoted to lattice morphisms. Let L be a lattice. A mapping f from L into L is called a *(lattice) endomorphism* if f preserves finite infima and suprema, i.e.,

$$f(x \vee y) = f(x) \vee f(y) \tag{2.5a}$$

$$f(x \wedge y) = f(x) \wedge f(y) \tag{2.5b}$$

for every $x, y \in L$. If, in addition, f is a bijection, then f is called an *automorphism*. In that case f^{-1}, the inverse mapping, also satisfies (2.5). Suppose that f is an automorphism. If L is a bounded lattice then $f(0) = 0$ and $f(1) = 1$. If, moreover, L is a Boolean lattice, then f also preserves complements:

$$f(x^*) = (f(x))^*. \tag{2.6}$$

Finally, if L is a complete lattice then the relations (2.5) remain valid for infinite suprema and infima:

$$f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$$

$$f(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} f(x_i).$$

Every mapping $f: L \rightarrow L$ satisfying at least one of the relations (2.5a), (2.5b) is increasing, i.e.,

$$x \leqslant y \Rightarrow f(x) \leqslant f(y). \tag{2.7}$$

The converse does not hold.

For future use we state the following lemma.

LEMMA 1. *Let L be a lattice with a zero, and let Λ be the (possibly empty) set of atoms. If f is an automorphism on L, then f leaves Λ invariant.*

PROOF. If Λ is empty then the lemma is trivially satisfied. So assume that $\Lambda \neq \varnothing$, and take $\xi \in \Lambda$. We must show that $f(\xi) \in \Lambda$. Assume that there is a $y \in L$ such that $y < f(\xi)$. Then $f^{-1}(y) < \xi$, hence $f^{-1}(y) = 0$. But this implies that $y = 0$. Thus $f(\xi)$ is an atom. □

2.2. Dilation and erosion

In this section we shall give an abstract definition of dilation and erosion on an arbitrary complete lattice. In Section 1.2 we have considered dilation and erosion on the complete Boolean lattice $\mathcal{P}(E)$, where E was $\mathbb{R}^n$ or $\mathbb{Z}^n$. We recall that two basic properties of dilation were:

(i) $(T_h X) \dot{\oplus} A \;=\; T_h(X \oplus A)$

(ii) $(\bigcup_{i \in I} X_i) \dot{\oplus} A \;=\; \bigcup_{i \in I} (X_i \oplus A)$,

where $T_h X = X_h$, i.e., T_h is translation along a vector $h \in E$. We note that the family of translations $\mathfrak{T} = \{T_h : h \in E\}$ forms a commutative group of automorphisms on the lattice $\mathcal{P}(E)$. Erosion is characterized by similar properties, the only difference being that in (ii) union has to be replaced by intersection. These two properties of dilation and erosion are used as the premises for an abstract definition. Assume for the remainder of this section that L is a complete lattice. Let $\mathfrak{T}$ be a commutative group of automorphisms on L. For notational convenience we shall write Tx instead of $T(x)$ if $T \in \mathfrak{T}$. A mapping $\psi : L \to L$ is called a *$\mathfrak{T}$-mapping* if ψ commutes with every T:

$$\psi(Tx) = T\psi(x), \quad T \in \mathfrak{T},\ x \in L.$$

We say that ψ is a *$\mathfrak{T}$-dilation* if

(i) ψ is a $\mathfrak{T}$-mapping

(ii) $\psi(\bigvee_{i \in I} x_i) \;=\; \bigvee_{i \in I} \psi(x_i)$,

for every index set I. Similarly, a mapping $\phi : L \to L$ is called a *$\mathfrak{T}$-erosion* if in (ii) the supremum is replaced by the infimum.

If L is a Boolean lattice, then the dual of a $\mathfrak{T}$-dilation is a $\mathfrak{T}$-erosion and conversely (the dual f^* of a mapping f on a complemented lattice is defined by $f^*(x) = (f(x^*))^*$). Let $\mathcal{A}$ be an arbitrary subset of $\mathfrak{T}$. It is easy to check that the mapping

$$\psi(x) \;=\; \bigvee_{T \in \mathcal{A}} Tx \tag{2.8}$$

is a $\mathfrak{T}$-dilation and that

$$\phi(x) \;=\; \bigwedge_{T \in \mathcal{A}} Tx \tag{2.9}$$

is a $\mathcal{T}$-erosion. Note that these expressions are nothing but straightforward generalizations of the original definitions: see Section 1.2. For the rest of this section we will restrict our attention to $\mathcal{T}$-dilations. It should be clear by now that dilation and erosion are just complementary notions. We address ourselves to the following question: is every $\mathcal{T}$-dilation of the form (2.8)? It turns out that we can give an affirmative answer to this question under two extra assumptions.

ASSUMPTION. L is atomic.

ASSUMPTION. For every couple $\xi,\eta\in\Lambda$ there is a $T\in\mathcal{T}$ such that $T\xi=\eta$.

If the latter assumption is satisfied, we call the automorphism group *total.* Now let $\psi:L\to L$ be a $\mathcal{T}$-dilation. Define $\mathcal{A}\subset\mathcal{T}$ by

$$T\in\mathcal{A} \Leftrightarrow T\xi\leqslant\psi(\xi),$$

where ξ is an arbitrary atom of L. By using that ψ is a $\mathcal{T}$-mapping and that $\mathcal{T}$ is total, one easily obtains that $\mathcal{A}$ is independent of the choice of ξ. Furthermore, one gets immediately that

$$\vee_{T\in\mathcal{A}}T\xi \leqslant \psi(\xi), \quad \xi\in\Lambda.$$

We can even show equality. Suppose, namely, that we have strict inequality. Then, since L is atomic, there exists an atom η such that $\eta\leqslant\psi(\xi)$ but not $\eta\leqslant\vee_{T\in\mathcal{A}}T\xi$. From the fact that $\mathcal{T}$ is total we know that $\eta=T'\xi$ for some $T'\in\mathcal{T}$. Hence $T'\xi\leqslant\psi(\xi)$, yielding that $T'\in\mathcal{A}$. But this implies that $\eta\leqslant\vee_{T\in\mathcal{A}}T\xi$, a contradiction. Thus we have proved that

$$\vee_{T\in\mathcal{A}}T\xi = \psi(\xi), \quad \xi\in\Lambda.$$

But now we are almost done. Consider namely an arbitrary element x of L. Then $x=\vee_{\xi\leqslant x}\xi$. Thus

$$\psi(x) = \psi(\vee_{\xi\leqslant x}\xi) = \vee_{\xi\leqslant x}\psi(\xi) = \vee_{\xi\leqslant x}\vee_{T\in\mathcal{A}}T\xi$$
$$= \vee_{T\in\mathcal{A}}\vee_{\xi\leqslant x}T\xi = \vee_{T\in\mathcal{A}}T(\vee_{\xi\leqslant x}\xi) = \vee_{T\in\mathcal{A}}Tx.$$

This proves the main result of this section.

THEOREM 1. *Let L be a complete, atomic lattice and let $\mathcal{T}$ be a total commutative group of automorphisms on L. Then every $\mathcal{T}$-dilation ψ is of the form*

$$\psi(x) = \vee_{T\in\mathcal{A}}Tx.$$

We can state a similar result for erosions on dually atomic lattices. Notice that $\mathcal{A}$ is the analogue of the structuring element of Section 1.2. It is time to give some examples.

EXAMPLES

(a) Consider the complete atomic (and dually atomic) Boolean lattice $\mathcal{P}(E)$, where E is $\mathbb{R}^n$ or $\mathbb{Z}^n$. Let $\mathfrak{T}$ be the group of all translations T_h, $h \in E$. Then every $\mathfrak{T}$-dilation is of the form

$$\Psi(X) = \bigvee_{T \in \mathcal{A}} TX,$$

or equivalently,

$$\Psi(X) = \bigcup_{h \in A} T_{-h}X = \bigcup_{h \in A} X_{-h} = X \oplus A,$$

where $A \subset E$ is given by: $h \in A$ if and only if $T_{-h} \in \mathcal{A}$. So in this case the class of all $\mathfrak{T}$-dilations (and of course of $\mathfrak{T}$-erosions) coincides with the original class.

In exactly the same way we obtain a complete characterization of $\mathfrak{T}$-dilations on the complete, atomic lattice $\mathcal{F}(\mathbb{R}^n)$, where $\mathfrak{T}$ is again the translation group. In this case every $\mathfrak{T}$-dilation Ψ is of the form

$$\Psi(X) = \overline{\bigcup_{h \in A} X_{-h}}.$$

By duality, we also get a complete characterization of $\mathfrak{T}$-erosions on the complete, dually atomic lattice $\mathcal{G}(\mathbb{R}^n)$.

(b) An advantage of our approach is that we are free to choose any automorphism group we want to: it is only required that this group is commutative and total. An interesting example is provided by the rotation-multiplication group. Consider the complete, atomic Boolean lattice $\mathcal{P}(\mathbb{C} \setminus \{0\})$. Let '$\cdot$' be the complex multiplication on $\mathbb{C}$. Let T_z be the automorphism given by

$$T_z X = \{x \cdot z : x \in X\}.$$

If $z = re^{i\theta}$ in polar coordinates, then T_z can be interpreted as a combination of a rotation by an angle θ and a multiplication with factor r. Then $\mathfrak{T} = \{T_z : z \in \mathbb{C} \setminus \{0\}\}$ forms a commutative automorphism group which is total. Needless to say that the performance of dilation and erosion in this example is completely different from the classical situation. In the discrete case a polar grid is required: see Figure 6 below.

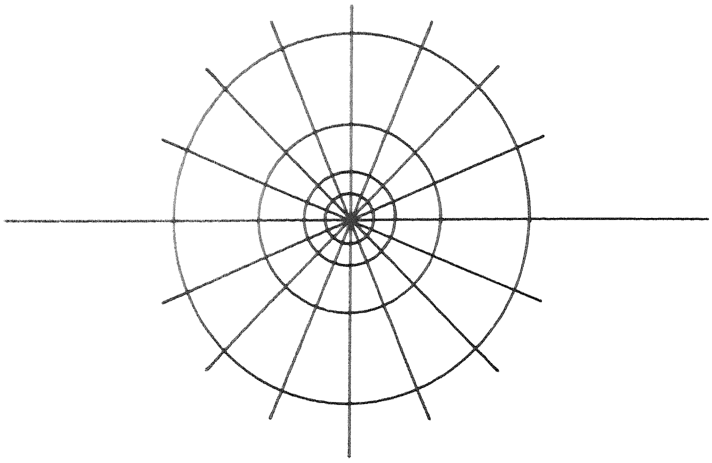


FIGURE 6. The polar grid of example (b)

(c) As a final example, we mention the following variant of example (b). Consider the Boolean lattice $\mathcal{P}(\mathbb{C})$ and the total commutative automorphism group $\mathcal{T} = \{T_z : z \in \mathbb{C}\}$ defined by

$$T_z X = \{x \cdot z : x \in X\},$$

where $z_1 \cdot z_2 = (r_1 + r_2)e^{i(\theta_1 + \theta_2)}$, if $z_j = r_j e^{i\theta_j}$.

2.3. *Increasing transformations and Matheron's theorem*

From Matheron's theorem we learned that every increasing translation invariant transformation on $\mathcal{P}(E)$, where E is $\mathbb{R}^n$ or $\mathbb{Z}^n$, can be written as an intersection of dilations, or, alternatively, as a union of erosions. In the present section we will show that this result can be established within our framework. But before stating and proving this generalization, we present an alternative formulation of the results obtained in the previous section. Actually, this reformulation is suggested by the examples above. In these examples the automorphism group $\mathcal{T}$ is isomorphic with a group structure on Λ, the set of all atoms. This is no coincidence but just an alternative formulation of our second assumption. To see this, assume that L is a complete atomic lattice, and that $\mathcal{T}$ is a total commutative automorphism group on L. First we note that, if for some $T_0 \in \mathcal{T}$ and some $\xi \in \Lambda$ we have $T_0\xi = \xi$, then this holds for every $\eta \in \Lambda$, which amounts to saying that T_0 is the identity mapping. Suppose namely that $\eta \in \Lambda$. Then there is a $T \in \mathcal{T}$ so that $\eta = T\xi$. Hence $T_0\eta = T_0 T\xi = TT_0\xi = T\xi = \eta$.

Now fix an arbitrary $\omega \in \Lambda$. We call ω the origin. For every $\xi \in \Lambda$ there exists a $T_\xi \in \mathcal{T}$ such that $T_\xi\omega = \xi$. Thus we can define an operation $+$ on Λ as follows:

$$\xi + \eta = T_\xi T_\eta \omega, \quad \xi, \eta \in \Lambda.$$

This definition makes sense because it is independent of the particular choice of T_ξ. It is easy to see that $(\Lambda, +)$ is a commutative group with identity ω. Conversely, every commutative group operation $+$ on Λ 'generates' a total commutative automorphism group on L. Let $-\xi$ denote the inverse of ξ with respect to the group operation $+$.

It should be clear by now how one can rewrite (2.8) and (2.9) in terms of the group operation $+$. Let ψ be given by (2.8) and define $a \in L$ by: $a = \bigvee_{T\in\mathfrak{T}} T^{-1}\omega$. Then

$$\psi(x) = x\dot{\oplus}a := \bigvee_{\alpha\leqslant a} x_{-\alpha} = \bigvee\{\xi\colon a_\xi \wedge x \neq 0\}, \quad x \in L.$$

Here $x_\alpha = \{\xi+\alpha\colon \xi \leqslant x\}$. Similarly the $\mathfrak{T}$-erosion of (2.9) can be written as:

$$\phi(x) = x\check{\ominus}a := \bigwedge_{\alpha\leqslant a} x_{-\alpha} = \bigvee\{\xi\colon a_\xi \leqslant x\}, \quad x \in L.$$

Before we give the abstract version of Matheron's theorem, we recall that a mapping $f: L \to L$ is increasing if $x \leqslant y$ implies that $f(x) \leqslant f(y)$.

THEOREM 2. *Let L be a complete, atomic lattice, and let $f: L \to L$ be an increasing $\mathfrak{T}$-mapping, then*

$$f(x) = \bigvee_{a\in\mathcal{V}} (x\check{\ominus}a),$$

where $\mathcal{V} = \{a \in L\colon \omega \leqslant f(a)\}$ is the kernel of f.

PROOF. We show that $\xi \leqslant f(x)$ if and only if $\xi \leqslant \bigvee_{a\in\mathcal{V}} (x\check{\ominus}a)$.

(i) Let $\xi \leqslant f(x)$. Then $\omega \leqslant T_{-\xi} f(x) = f(T_{-\xi} x)$. Hence $y := T_{-\xi} x \in \mathcal{V}$ by definition. Thus

$$\bigvee_{a\in\mathcal{V}} (x\check{\ominus}a) \geqslant x\check{\ominus}y = y_\xi \check{\ominus} y = \bigwedge_{\eta\leqslant y} (y_\xi)_{-\eta}$$

$$= \bigwedge_{\eta\leqslant y} (y_{-\eta})_\xi = T_\xi\left(\bigwedge_{\eta\leqslant y} y_{-\eta}\right) \geqslant T_\xi \omega = \xi.$$

(ii) Conversely, assume that $\xi \leqslant \bigvee_{a\in\mathcal{V}} (x\check{\ominus}a)$. So there is an element $a \in \mathcal{V}$ such that

$$\xi \leqslant x\check{\ominus}a = \bigwedge_{\eta\leqslant a} x_{-\eta}.$$

Therefore $\xi \leqslant x_{-\eta}$ for every η satisfying $\eta \leqslant a$. But this implies that $\eta \leqslant x_{-\xi}$, for every $\eta \leqslant a$. Thus

$$a = \bigvee_{\eta\leqslant a} \eta \leqslant x_{-\xi},$$

and by the increasingness of the mapping f, $f(a) \leqslant f(x_{-\xi})$. Since $a \in \mathcal{V}$, we find that $\omega \leqslant f(a) \leqslant f(x_{-\xi})$, hence $\xi = T_\xi \omega \leqslant T_\xi f(x_{-\xi}) = f(x)$, which proves the result. □

Similarly one can show that on a complete, dually atomic lattice every increasing transformation can be written as the infimum of $\mathfrak{T}$-dilations. On a complete, atomic Boolean lattice both characterizations hold.

2.4. Concluding remarks
As a first step towards an abstract algebraic approach, the results obtained so far may seem quite satisfactory. However, as we will indicate below, a lot remains to be done. But let us first give a brief summary of our results.

If L is a complete, atomic lattice, e.g. $L=\mathcal{F}(\mathbb{R}^n)$, then every $\mathcal{T}$-dilation ψ is of the form

$$\psi(x) = x\oplus a, \quad x\in L,$$

for some $a\in L$. Furthermore, every increasing $\mathcal{T}$-mapping is a supremum of $\mathcal{T}$-erosions. Similarly, if L is a complete, dually atomic lattice, e.g. $L=\mathcal{G}(\mathbb{R}^n)$, then every $\mathcal{T}$-erosion takes the form

$$\varphi(x) = x\ominus a, \quad x\in L,$$

for some $a\in L$, and every increasing $\mathcal{T}$-mapping is an infimum of $\mathcal{T}$-dilations. These results become more transparent if L is a complete Boolean lattice. In that case, the assumption that L is atomic is equivalent to the assumption that L is dually atomic, and $\mathcal{T}$-dilations and $\mathcal{T}$-erosions are dual mappings. We recall that a complete, atomic Boolean lattice L is isomorphic with the field $\mathcal{P}(A)$, where A is the set of all atoms, and that every total, commutative automorphism group on L induces a group structure on A. Thus, algebraically speaking, there is no distinction between this case and the original case described in Part 1 where $L=\mathcal{P}(\mathbb{R}^n)$; see also Section 2.2, Example (a).

Let L be a complete lattice and let $\mathcal{T}$ be a total, commutative automorphism group on L. We define $M^+_{\mathcal{T}}(L)$ as the set of all increasing $\mathcal{T}$-mappings on L. Besides $\mathcal{T}$-dilations and $\mathcal{T}$-erosions, this set also contains compositions of these transformations such as $\mathcal{T}$-closings and $\mathcal{T}$-openings. On the set $M^+_{\mathcal{T}}(L)$ we can define the partial order $\leqslant$ by:

$$f\leqslant g \iff \forall_{x\in L}: f(x)\leqslant g(x).$$

Then $M^+_{\mathcal{T}}(L)$ becomes a complete lattice with supremum and infimum respectively given by

$$(f\vee g)(x) = f(x)\vee g(x), \quad x\in L$$

$$(f\wedge g)(x) = f(x)\wedge g(x), \quad x\in L.$$

These observations imply that every $\mathcal{T}$-mapping which is obtained from $\mathcal{T}$-dilations and $\mathcal{T}$-erosions by means of suprema, infima, and compositions is increasing; no such thing as the hit-or-miss transformation can be obtained in this way. If, however, L is a Boolean lattice then the mapping $x\to f(x^*)$ is a decreasing $\mathcal{T}$-mapping if $f\in M^+_{\mathcal{T}}(L)$. Replacing the complement by the pseudo-complement, we can do the same trick if L is a Brouwerian lattice.

In Section 1.3 we have argued, following Serra, that a theory of transformations is not very meaningful if one cannot give shape to the notion of (semi-) continuity, which requires a topology on the lattice L. In the second part of this paper we have consistently omitted to speak about topological aspects. Here we shall somewhat retrieve this omission by mentioning in a few lines an important result that can be found in the literature. We will certainly come back to this point in the future. It needs no explanation that a topology on L has to be related to the ordering relation, and that the automorphism group $\mathcal{T}$ should have the right continuity properties with respect to this topology.

The lattice $\mathcal{F}(\mathbb{R}^n)$ of all closed subsets of $\mathbb{R}^n$ with the opposite ordering ($X \leqslant Y$ if $Y \subset X$) is a so-called *continuous lattice*. On a continuous lattice one can define the so-called *Lawson topology*. On the lattice $\mathcal{F}(\mathbb{R}^n)$ this topology coincides with the hit-or-miss topology (see [5] for more details). This observation which we consider to be an extra justification of our approach, may serve as an underlining of the assertion that thinking about mathematical generalizations is not only a pleasant pass-time (it is, of course), but may also give a deeper understanding of the original theory.

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